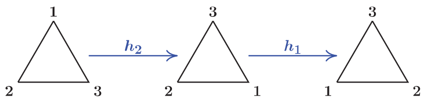

Figure 2.9: Graphical depiction of the fact that $h_1 \circ h_2 = t_{2\pi/3}$.

For the rest of the multiplication table see Example 2.3.7 below.

It is not immediately obvious, but the symmetry groups for all regular polygons (equilateral triangle, square, regular pentagon, etc.) are closely related. They share many of the same major structures and form a nice family of finite groups which will be relatively straightforward to describe in general. For example, regardless of which polygon we choose, it will always be symmetric across any angle bisector, thus reflections across the angle bisectors will always account for a subset of the symmetries. Similarly, the regularity of the polygons also implies that we may always rotate around the center of the polygon – provided that we rotate by an angle measurement which takes each vertex to another vertex. In general, the group of symmetries of a regular polygon is called a *Dihedral group* and thus, we may consider the family of Dihedral groups corresponding to the family of regular polygons.

Definition 2.3.6 (The Dihedral Groups, D_n: Version 1)**.**

The group $D_n = E(P_n)$ of symmetries of the regular n-sided polygon P_n is called the Dihedral Group. It is a non-Abelian group of order $2n$ containing rotations around the center of P_n, $\{\text{Id}, t_{2\pi/n}, t_{4\pi/n}, \ldots, t_{(2n-2)\pi/n}\}$, and reflections through each angle bisector and each perpendicular bisector of the sides, $\{h_1, h_2, \ldots, h_n\}$.
(Note: When n is odd, each angle bisector is simultaneously a perpendicular bisector of the side opposite the angle and when n is even, angle bisectors bisect opposite pairs of angles.)

The equilateral triangle is the regular polygon P_3 and hence $E(P_3)$ corresponds to the Dihedral group, D_3. As promised, we now give the full multiplication table for D_3.

Example 2.3.7 (Multiplication Table for D_3)**.**
Here is the full table using the elements described in Example 2.3.5.

$\circ$	Id	$t_{2\pi/3}$	$t_{4\pi/3}$	h_1	h_2	h_3
Id	Id	$t_{2\pi/3}$	$t_{4\pi/3}$	h_1	h_2	h_3
$t_{2\pi/3}$	$t_{2\pi/3}$	$t_{4\pi/3}$	Id	h_3	h_1	h_2
$t_{4\pi/3}$	$t_{4\pi/3}$	Id	$t_{2\pi/3}$	h_2	h_3	h_1
h_1	h_1	h_2	h_3	Id	$t_{2\pi/3}$	$t_{4\pi/3}$
h_2	h_2	h_3	h_1	$t_{4\pi/3}$	Id	$t_{2\pi/3}$
h_3	h_3	h_1	h_2	$t_{2\pi/3}$	$t_{4\pi/3}$	Id

Example 2.3.8 (Symmetries of a Regular Octagon, D_8)**.**
The definition of the Dihedral group claims that there should be exactly 16 elements in D_8. Let's quickly verify that count and describe all of the elements. Recall that any symmetry must send the vertices of the octagon back to vertices and must actually maintain adjacency, meaning that if two vertices were next to each other around the octagon, then their images must be next to each other as well. With that in mind, we can get an upper bound on the number of symmetries. First we choose where one vertex goes (8 choices), then we decide where an adjacent vertex goes. Since it must remain adjacent there are two choices. After that, every vertex is forced into a location by the adjacency criterion. Hence there are $8 \cdot 2 = 16$ potential maps. Now we just need to demonstrate 16 symmetries that are distinct from one another, see Figure 2.10.

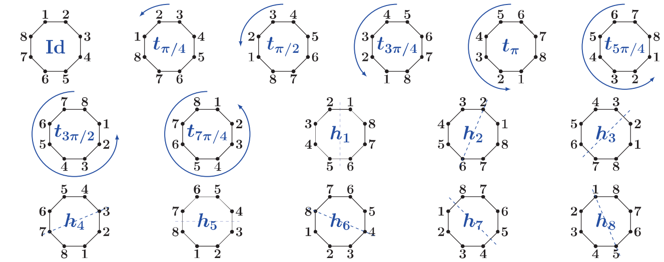

Figure 2.10: Graphical depiction of the 16 symmetries in D_8.

Of course, the choice of labeling for the vertices is completely arbitrary (and unimportant to the definition of the group). In addition, the naming of the 8 distinct reflections is also an arbitrary choice. In fact, in many ways, these descriptions/labelings hide some of the most beautiful structures that all dihedral groups share. They can also be quite cumbersome when it comes to calculating products within these groups as we, in principle, need to keep pictures of labeled polygons in our head (or on paper!) to manipulate with each symmetry. Thankfully there is a much better way to describe the dihedral groups:

Definition 2.3.9 (The Dihedral Groups, D_n: Version 2)**.**

The Dihedral Group, D_n, is a non-Abelian group of order $2n$ containing "rotations," $\{\text{Id}, t, t^2, \ldots, t^{n-1}\}$, and "reflections," $\{h, ht, \ldots, ht^{n-1}\}$, which multiply according to the following rules: $t^n = \text{Id}$, $h^2 = \text{Id}$, and $th = ht^{-1}$, in addition to normal exponent rules.
(These rules imply that $t^k h = ht^{-k}$ for all k as well)

Observe that the group operation rules described above give us a way to take any product of powers of h and t and rewrite it in the form of one of the elements listed. Most of that process is pretty straightforward other than the interesting rule which tells us how to rewrite elements that have an h on the right instead of the left. For example, suppose we were in D_4 and we wanted to multiply t^3 and ht. So we would like to know which element in our list is represented by t^3ht. The rules imply that we can rewrite t^3h as $h\left(t^3\right)^{-1} = ht$. Hence, using associativity we have $t^3ht = \left(t^3h\right)t = (ht)t = ht^2$ and we've now written this product in the proper form.

In a lot of ways, this second description of the dihedral group makes multiplication completely intuitive. See for example, the new multiplication table for D_3 in Example 2.3.10 below. The cost here is that, in some ways, it is harder to make sense of the specific geometry involved. Moreover, at the moment we have no way of knowing whether this different group we've described is really the "same" as the original (with just a renaming of the elements). In Exercise 79, you'll show that the dihedral group from the Version 1 definition satisfies all of the rules in the Version 2 definition, but we won't be able to show that these groups are actually "the same" (what we call *isomorphic*) until we get to Chapter 5.