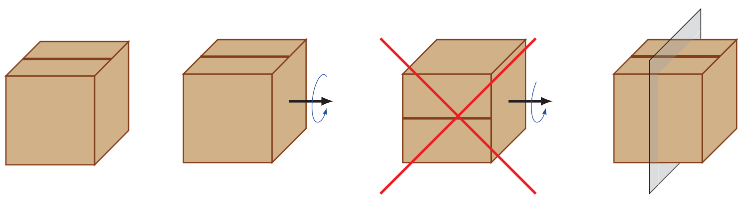
2.3 Groups of Symmetries

One really natural place to find groups is when looking at the symmetries of an object. As I alluded to back in Section 2.1, symmetries of a geometric object correspond to a special subset of the set of bijections on that object. We'll see exactly what extra conditions need to be satisfied shortly, but first we should tie the mathematical notion of symmetry to the one you are already quite familiar with in the natural world. Our brains are quite good at detecting that kind of symmetry in the real world. In fact, symmetry is often associated with beauty in various art forms as well. But, what is it, exactly, that we mean in those settings when we claim that an object has "symmetry?" One definition states that objects have *symmetry* if they "contain parts that can be interchanged without changing the whole."

The idea is that parts of the whole are so physically similar that we could imagine being unable to tell the difference if those parts were swapped. Imagine a perfectly cubic cardboard box, centered at the origin in $\mathbb{R}^3$, that is completely devoid of any printing or writing on its surface. Based on the way these boxes are created, the box can be opened in two different places that are directly opposite each other, see Figure 2.2. (Note that I've chosen to take the positive x-axis as coming out of the page, the positive y-axis as heading right horizontally, and the postive z-axis as pointing up vertically as is frequently done in multi-variable calculus courses.)

The box looks exactly the same if we were to rotate it 180 around the x-axis, or the y-axis, or the z-axis (you'll have to trust me that the second box has actually been rotated because, of course, we can't tell!). Notice, however, that if we only rotate 90 degrees around either of those axes then we've definitely changed what the box looks like, thus the box does not have this kind of symmetry. An object that looks identical after a rotation of some kind is said to have *rotational symmetry*.

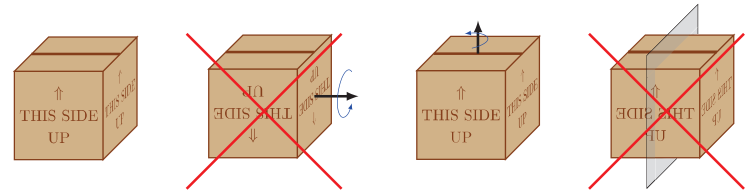
Similarly, we can see that our cardboard box has two identical halves if we were to cut it down the middle, using the xy-plane or the xz-plane or even the yz-plane (actually there are several other planes that would work as well). What we mean by identical here is that the two sides are exactly mirror images (or reflections!) of each other. An object that looks identical after a reflection through some plane (or through a line if its a 2-dimensional object) is said to have *reflective symmetry*.



Rotated 180° around y-axis. Rotated 90° around y-axis. Reflected across the xz-plane.

Figure 2.2: Cardboard box symmetries.

Observe that if we were to add identical text to each of the sides of the box that tell us which opening is the top, then the symmetries change. Now, a 180° rotation around the y-axis will not give us the same box back (of course, neither will a 90° rotation around the y-axis). However, a 180° rotation around the z-axis will work just fine, see Figure 2.3. These side labels also ruin the reflective symmetry that we had above because reflections now have the effect of changing how the text looks.



Rotated 180° around y-axis. Rotated 180° around z-axis. Reflected across the xz-plane.

Figure 2.3: Labeled cardboard box symmetries.

From a mathematical perspective, we can make the notion of symmetry more precise. To do so, we'll think of any objects as subsets of real space $\mathbb{R}^n$. What we are looking for then, are transformations that we can apply to real space that will maintain the features of that space while simultaneously giving us (what appears to be) the same object back. Such transformations we will call *symmetries* of the object. Thus, a 180° rotation around the y-axis is a symmetry of the unlabeled cardboard box, while the 90° rotation is not. Similarly, a reflection in the xz-plane is a symmetry of the unlabeled box, but is not a symmetry of the labeled one.

Definition 2.3.1 (Symmetry).

Given a subset $S \subseteq \mathbb{R}^n$, a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetry of S if:

- f preserves distance in $\mathbb{R}^n$,
- f maps the set S back to itself, i.e. $f(S) = S$.

We let $E(S)$ denote the set of all symmetries of S.

Important: It is the second condition, that $f(S) = S$, that makes symmetries bijections when restricted to S, i.e. $f|_S \in \text{Bij}(S)$ for all $f \in E(S)$. Certainly the identity map (dropping the subscript for simplicity) $\text{Id} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\text{Id}(x) = x$ for all $x \in \mathbb{R}^n$ is a symmetry of S (you should check that it satisfies both parts of the definition). However, just as in the case of bijections, the fact that $f(S) = S$ **as a whole**, does NOT mean that $f(x) = x$ for all $x \in S$ (in other words, there **do** exist other symmetries than just Id in most cases). The map f might rearrange all of the points in S, it just has to be the case that after any rearranging, we will have the entire set S covered.

Distance Preserving Maps

Maps that satisfy just the first criterion in Definition 2.3.1 are known as *distance preserving* maps. To expand on that notion, a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to preserve distance in $\mathbb{R}^n$ if for all pairs of points $\vec{x} = (x_1, x_2, \ldots, x_n), \vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ we have

$$\text{dist}(\vec{x}, \vec{y}) = \text{dist}(f(\vec{x}), f(\vec{y})).$$

That is, the distances between any pairs of points have to be maintained by application of the map f, so that the distance between any two points is exactly equal to the distance between their two images under f.