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Introduction

The content of this work is meant to be used as a one semester introduction to Abstract Algebra via Group Theory for mathematics students in a liberal arts setting. With that in mind, I have made the unorthodox decision to write this book in a much more familiar style than one would typically (read: effectively always) see in a mathematics journal article or textbook. Not only is there heavy usage of the familiar "you" rather than addressing the more formal and mysterious "reader," but there are also many parenthetical notes (typically in blue like this) and warnings in red meant to help clarify difficult portions of the text, anticipate common errors, or offer alternative ways of interpreting the definitions, examples, and results. The hope is that making the text easier to read and more colloquial will help encourage all of you (the students) to read it!

The overarching goal of the book is two-fold. First, I would like you all to become familiar with the notion of groups and to gain the interest and motivation to look for groups everywhere. Essentially, any time you find a set that is closed under some nice operation I would like you to wonder whether that set (together with the operation) forms a group. Groups are somewhat fundamental objects that appear all over the place. Not just in the abstract realm of mathematics, but also in more exotic places like molecular biology and physics and probably many more places where they have yet to be noticed. One of the major aims (and strengths) of mathematics is to find generalizations that allow us to see the connections between many different-looking topics. Once the topics have been tied together through a common generalization, then the tools from each discipline become available to the other. I would love to see each of you go out into the world and find more of these kinds of connections!

With that goal in mind, the second aim really piggy-backs on the first – as we begin to understand groups, a very natural question to ask is: "What kinds of groups actually exist?" What does it mean for two groups (that look different) to be the same? In other words, suppose that – after working through this book and perhaps a related course – you find yourself searching for groups in your future area of work. One day, while working and searching you discover some group "in the wild" so to speak. How will you know which group it is? Or, more pertinently, how will you be able to use your knowledge of groups gained here to help you gain information about your wild group? If you can successfully identify the group you've found with another one that you already understand, then you'll be able to take everything that you've learned here and apply it to your new setting. So we'd like to both gain knowledge of many different common examples of groups and also learn how to relate two groups to one another via maps between them.

Topics Covered

The text begins with a significant introductory chapter in which we cover many topics that you may have seen before (introductory set theory, number theory, and induction) in the hopes that all of you will approach the group theoretic sections with a similar foundation. Chapter 2 introduces the notion of groups via examples which you should already be familiar with (although probably not from this point of view). We then prove many nice properties that all groups share before starting to delve into the properties of individual elements and nice subsets (called subgroups). Chapter 3 contains some important examples of infinite families of groups – the cyclic groups and the symmetric groups – and then discusses a way of combining known groups to create new ones.

Chapter 4 discusses subgroups in further depth. Specifically introducing cosets of subgroups and normal subgroups with an aim toward proving Lagrange's Theorem and defining quotient groups. Chapter 5 then circles back to maps between sets with the additional motivation that we would like those maps to respect groups structures. Maps that do so are called group homomorphisms and they allow us to begin seeing connections between groups that "look" very different. The notion of isomorphisms (group homomorphisms that are also bijections) will then allow us to actually identify groups as being effectively "the same" and will allow us to begin classifying small finite groups. This chapter finishes off with some nice structure theorems relating groups to their images under homomorphisms and finally Cayley's Theorem which gives us a way of understanding all finite groups in terms of the family of symmetric groups. From my perspective, this would serve as a nice introduction to Representation Theory as well.