In fact, we can also adjust our well-ordered sets by only considering certain types of numbers, thereby affecting criteria (ii) in the corresponding induction arguments. For example, if we find/conjecture that some property holds for all even natural numbers, then we can still formulate an induction argument. As a subset of $\mathbb{N}$, the even numbers, $2\mathbb{N}$, are also well-ordered. For an induction argument, we can either directly adjust the entire Theorem – in this case changing the base case to $k = 2$, changing critera (ii) to say that "if $k \in S$, then $k + 2 \in S$ too," and changing the conclusion to $S = 2\mathbb{N}$ – OR, we can replace the even number placeholders in our desired property with $2k$ placeholders instead, so that k will run through all of $\mathbb{N}$ as $2k$ runs through all even natural numbers. In the following example, our stated property only holds for powers of 2, hence we will adjust our induction argument to match using the second method.

Example 1.2.8. [Triominoes]
A *triomino* consists of three squares connected in an L-shape. Just as we might try to tile a chessboard with dominoes (see Exercise 3), we could also try to tile one with triominoes instead. Of course, since triominoes cover three squares at a time, it is impossible to completely cover any grid unless is has a number of squares that is divisible by 3. As an example, we demonstrate a tiling of the 4×4 grid that has had the top left corner removed using 5 triominoes, see Figure 1.5.

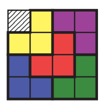

Figure 1.5: A tiling with triominoes of a 4×4 grid with one square removed.

<u>Claim:</u> A $2^n \times 2^n$ grid with any single square removed can be successfully covered by triominoes for any $n \in \mathbb{N}$.

We will prove this fact by induction. The base case is a 2×2 grid with a single square removed (the case when $n = 1$). With a 2×2 grid, no matter which square is removed, the remaining three can together be covered by a single triomino. Hence, our claim is true in the base case.
For the induction step, let $k \geq 1$ and assume that a $2^k \times 2^k$ grid with any single square removed can be covered by triominoes. Observe then, that any $2^{k+1} \times 2^{k+1}$ grid can be split up into four $2^k \times 2^k$ grids, but only one of those subgrids will contain the square that has been removed. Without loss of generality, we may assume that the top left subgrid contains the missing square (by rotating if necessary). The key idea here is that if we place a triomino near the center so that it covers one square in each of the other subgrids (see Figure 1.6), then we will have successfully split a $2^{k+1} \times 2^{k+1}$ grid into four $2^k \times 2^k$ subgrids *each with a single square removed*. From our induction assumption, we can tile each of those subgrids, hence together we can tile the entire $2^{k+1} \times 2^{k+1}$ grid. This completes the induction step, thus, by induction, we can tile a $2^n \times 2^n$ grid with one square removed for any $n \in \mathbb{N}$.

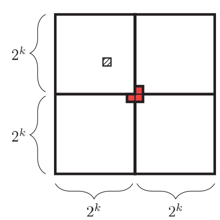

Figure 1.6: Breaking the $2^{k+1} \times 2^{k+1}$ grid down into four $2^k \times 2^k$ subgrids.

Perhaps more interestingly, we can also adjust criteria (ii) in another way to get a different form of induction. In some cases, the desired statement when $n = k + 1$ may very naturally relate to multiple cases with $n \leq k$. Thus, to show that a statement holds when $n = k + 1$ (or equivalently, showing that $k + 1$ is in the subset S) it may be helpful to know that the statement holds for both $n = k$ and $n = k - 1$ or, more generally, for all $n \leq k$. This second form of induction where we assume that $n \in S$ for all $n \leq k$ is often called *strong induction*.

Theorem 3 (Strong Induction)**.**

Let $S \subseteq \mathbb{N}$ be a subset of the positive integers. If S satisfies

(i) $1 \in S$

(ii) Whenever $1, 2, \ldots, k \in S$, then $k + 1 \in S$ too.

then $S = \mathbb{N}$.

Proof. Let $T = \mathbb{N} \setminus S$ be all of the elements of $\mathbb{N}$ that are not in S. If $T = \varnothing$ then we are done, so we assume (towards a contradiction) that $T \neq \varnothing$. Since $T \subseteq \mathbb{N}$ and non-empty, the Well-Ordering Principle implies that there exists some smallest element $t \in T$. By assumption (i) we know that $1 \in S$, thus $1 \notin T$ and it follows that $t > 1$. Now $1, 2, \ldots, t - 1$ are positive and must be in S (since t was minimal), but then by assumption (ii) we have $(t - 1) + 1 = t \in S$ which is a contradiction. Thus T must have been empty and $S = \mathbb{N}$ as claimed. □

The assumptions for the two types of induction definitely feel different, but the proofs of their validity are almost exactly the same. On first reading, Strong Induction requires much stronger assumptions and thus appears to be more difficult to apply in general. Certainly any statement that can be proven using induction can also be proven using strong induction (where we appear to assume more). From that perspective it is quite surprising that these two statements are actually completely equivalent. That is, any statement that can be proven using induction can be proven with strong induction **and vice versa!**

We omit the proof of this fact here since it does not add to the discussion, but the important idea to take away is that either form of induction may be used and we should always choose the version which fits most naturally with the statements we are trying to prove. That is to say, if the case when $n = k + 1$ depends naturally on the case when $n = k$, then we will use standard induction, while if more cases are needed, we will opt for strong induction as in the next example.

An important point to emphasize here is that we won't know which type of induction or what sorts of base case(s) we will need until we work out an inductive argument. In practice, we should first attempt to workout how later cases depend on previous ones. Next, we should determine which cases can be covered by our inductive argument. And finally we should prove all earlier cases directly as base cases. In the following argument we will attempt to demonstrate this process before writing down a formal proof.

Example 1.2.9.
Suppose that we want to prove that for any integer $m > 1$ we have $F_{m-1}F_n + F_mF_{n+1} = F_{m+n}$ for all $n \in \mathbb{N}$. If we want to create a general inductive argument then we will have to relate the case when $n = k + 1$ to earlier cases. Observe that when $n = k + 1$ the left hand side (LHS) of our statement is $F_{m-1}F_{k+1} + F_mF_{k+2}$. The definition of the Fibonacci sequence naturally relates each term F_k to the two previous terms. If we apply that definition to both F_{k+1} and F_{k+2} we obtain $F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)$. We can rearrange this to look more like our original expression if we distribute and then split the terms up. For example, $F_{m-1}F_k$ should be paired up with F_mF_{k+1} if we are to match the LHS of the original statement. Thus, after rearranging we have $(F_{m-1}F_k + F_mF_{k+1}) + (F_{m-1}F_{k-1} + F_mF_k)$.