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CAYLEY GRAPHS OF SYMMETRIC GROUPS GENERATED BY REVERSALS

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1. Introduction. An important measure of connectivity in a graph is its expansion constant. A family of k-regular graphs is called an *expander* if this constant is uniformly bounded away from zero. Expanders have many important applications in computer science (we refer the reader to [4] or [3] for a good list), and what one typically needs for these applications is an explicit construction. One of the standard methods for constructing expanders uses Cayley graphs of finite groups (for example, the "Ramanujan graphs" constructed by Lubotzky, Phillips, and Sarnak [5]). With regard to expander problems, Lubotzky poses an interesting question (Problem 10.3.4 in [4]). He asks whether Cayley graphs of the symmetric groups S_n could contain a family of expanders. In attempting to answer this question, generating sets consisting of involutions are a natural place to begin. The special case of Cayley graphs generated by transpositions has already been well studied (see, e.g., [1] and [2]), but there can be no true expanders in such a set since the number of transpositions needed to generate S_n, and hence the regularity of the Cayley graph, grows with n. In this paper we consider Cayley graphs of S_n generated by "reversals". By definition a reversal is an involution that flips an entire interval. For example, the permutation 154326 is a reversal since it flips the interval 2345. In looking for expanders, reversals (at least on first glance) appear to be a better special case to study because there exist finite generating sets that do not grow with n (any S_n can be generated by just 3 reversals, Proposition 1). It turns out, however, that reversals do not do the trick. The main result of this paper is to show that no family of Cayley graphs of the symmetric group generated by reversals can be a family of expanders.

2. Notation and Terminology. Before we can begin to discuss our assertion we provide the necessary definitions, notation, and propositions. Let G be a group and T a subset of G. Then the associated *Cayley graph*, denoted by $X(G,T)$, is the graph whose vertices are the elements of G and whose edges correspond to pairs $(g_1, g_2) \in G$ such that $g_2 = g_1 t$ for some $t \in T$. Given a subgroup $H \subseteq G$, the associated *Schreier graph*, denoted $X(H\backslash G, T)$, is the graph (with loops) whose vertices are the right cosets $H\backslash G$ with an edge joining Hg to Hgt whenever $g \in G$ and $t \in T$. (The reader may be more familiar with the notation G/H for the set of cosets. However, we have adopted the convention that edges are defined by the right action of G on itself, hence the vertices of the Schreier graph will correspond to the set of right cosets, which we denote instead by $H\backslash G$.) *A priori* Cayley and Schreier graphs are directed graphs, however, for us T will be a set of involutions so we can assume that the graphs are undirected. (For any vertex pair $g_1, g_2 \in G$ the same involution $t \in T$ will both take g_1 to g_2 and g_2 to g_1; thus edges in the Cayley graph come in opposite pairs which we identify as a single undirected edge.) For us, G will be the symmetric group S_n. Given two elements $a, b \in S_n$, we follow the convention that ab is the composition $b \circ a$; for example, in cycle notation, $(123)(12) = (23)$. An involution t in S_n is a

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reversal if there exist i, j with $1 \leq i < j \leq n$, such that

$$t(p) = \begin{cases} \text{p} & \text{if } p < i \text{ or } p > j \\ \text{j+i-p} & \text{if } i \leq p \leq j. \end{cases}$$

We denote such a reversal by $t = [i, j]$. In this paper, we study Cayley graphs $X(S_n, T)$ where T consists of only reversals (and T generates S_n). For example, if we take $G = S_3$ and $T = \{[1,2],[2,3]\}$, the resulting Cayley graph is shown in Figure 1 (for the vertex labels, we use the one-line notation for permutations in S_3). The relevant Schreier graph for us will be the one corresponding to the subgroup

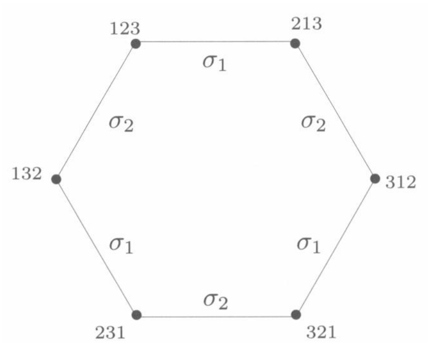

FIG. 1. *Cayley graph for* S_3 *with generators* $\sigma_1 = [1,2]$ *and* $\sigma_2 = [2,3]$

$H = S_2 \times S_{n-2}$. It can be shown that in such a Schreier graph the vertices correspond to 2-element subsets of $\{1, 2, 3, ..., n\}$. For example, if $n = 5$, the Schreier graph $X(H\backslash G, T)$ will have $\binom{5}{2}$ vertices, and the edges will depend on the particular choice of T. Figure 2 shows such an example with $T = \{[2,4],[1,4],[2,5]\}$. Given a graph X,

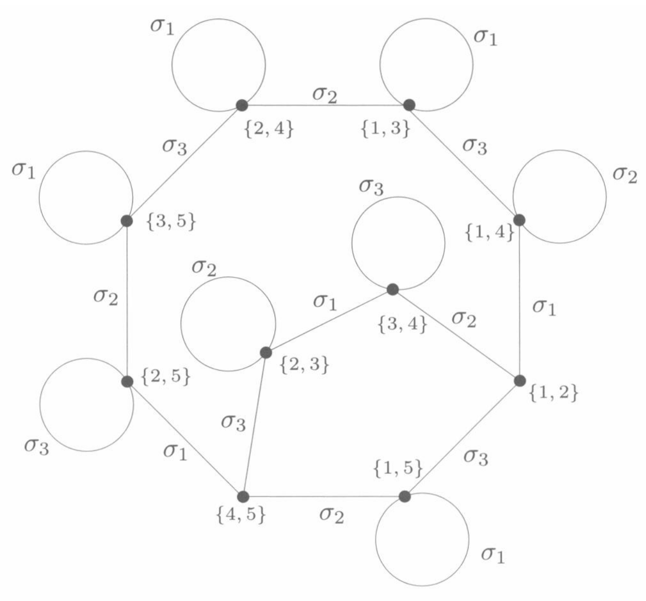

FIG. 2. *Schreier graph for* $S_2 \times S_2\backslash S_4$ *with generators* $\sigma_1 = [2,4]$, $\sigma_2 = [1,4]$, $\sigma_3 = [2,5]$

we let $V(X)$ denote the vertex set of X. If $A, B \subseteq V(X)$ form a partition of $V(X)$,

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we write $V(X) = A \uplus B$ and let $E(A, B)$ denote the set of edges joining elements of A to elements of B. The *expansion constant*, $\epsilon(X)$, of a graph X is defined by

$$\epsilon(X) = \inf_{A \uplus B = V(X)} \frac{|E(A,B)||V(X)|}{|A||B|}.$$

A family $\mathcal{F}$ of graphs is an *expander* (or *expanding family*) if there exists a constant $c > 0$ such that

$$c \leq \epsilon(X)$$

for all graphs X in the family $\mathcal{F}$. The expansion constant is a measure of the connectivity of a graph (the larger $\epsilon(X)$ is, the more highly connected X is). In most explicit constructions of regular families one usually sees this measure approach zero as one considers larger and larger graphs in the family. What makes an expander useful is that this measure of connectivity is bounded away from zero (the connectivity is above a certain measure) regardless of how large $X \in \mathcal{F}$ is. Therefore, we could use the construction for an expander to create highly connected networks for any arbitrarily large set of nodes. Note that the graphs in an expanding family must be connected (otherwise selecting A to be one component of X would result in $|E(A, B)| = 0$). In looking for an expanding family of Cayley graphs of the form $\mathcal{F} = \{X(S_n, T_n)\}$, then, this means that each set of reversals T_n must generate S_n. Restricting generating sets to transpositions cannot produce an expanding family since any set T of k transpositions fixes at least $n - 2k$ numbers, and therefore (for sufficiently large n) T cannot generate S_n. On the other hand, the following theorem shows that, at least in principle, one might find a family of expanders from Cayley graphs of S_n based on reversals.

PROPOSITION 1. *Any symmetric group S_n can be generated by a set of* 3 *reversals.*

Proof. Let a, b, and c be the reversals $a = [1, 2]$, $b = [1, n]$, and $c = [2, n]$. It is well known that the set of transpositions $P = \{(12), (23), ..., ((n-1)n)\}$ is a generating set for the symmetric group S_n, therefore it is sufficient to show that each element of P can be expressed as a product of the elements a, b, and c. By starting with a and alternating conjugations by b and c we eventually obtain all of the elements in P:

$$\begin{gathered}
(12) = a \\
((n-1)n) = bab \\
(23) = cbabc \\
((n-2)(n-1)) = bcbabcb \\
(34) = cbcbabcbc \\
\vdots
\end{gathered}$$

Thus, the set $\{a, b, c\}$ generates S_n. □ In fact, the number 3 is optimal, as the following Proposition shows.

PROPOSITION 2. *If T is a set of involutions that generates S_n (for $n > 3$), then* $|T| \geq 3$.

Proof. It suffices to show that no 2 involutions generate S_n for $n > 3$. We show this by considering the Cayley graphs of such cases. The Cayley graph for a 2-involution generating set has $n!$ vertices and is 2 regular. There is only one way to create a connected, 2-regular graph with $n!$ vertices and that is with a length $n!$ cycle. The two edges coming from each vertex are labeled by our two chosen involutions,

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respectively. Thus, by starting at any vertex we must be able to alternate the usage of our two involutions until we pass through every vertex in the graph and then return to our starting point. Therefore, the product of our two involutions must be of order $n!/2$. But for $n > 3$, S_n contains no element of order $n!/2$. □

3. The Main Theorem. In this section we prove our main theorem:

THEOREM 3. *Let $\mathcal{F} = \{X(S_n, T_n)\}$ be a family of Cayley graphs of symmetric groups such that for each n, T_n is a set of k reversals. Then $\mathcal{F}$ is* not *an expander.* To prove this theorem we need the following lemma which allows us to substitute the more manageable Schreier graphs for our Cayley graphs.

LEMMA 4. *If $\{X(G_n, T_n)\}$ is an expander and $H_n \subseteq G_n$ are subgroups, then $\{X(H_n\backslash G_n, T_n)\}$ is an expander.*

Proof. Let X be one of the Schreier graphs $X(H\backslash G, T)$ in our family and let $\widetilde{X}$ be the corresponding Cayley graph $\widetilde{X}(G, T)$. Then there is a natural map of graphs $\pi : \widetilde{X} \to X$ given on the vertices by the projection $G \to H\backslash G$. Given $A \subseteq H\backslash G$, we let $\widetilde{A} \subseteq G$ denote the inverse image $\pi^{-1}(A)$. Let $A \uplus B$ be a partition of $V(X)$ $(= H\backslash G)$. Letting m be the order of H, we have $|\widetilde{A}| = m|A|$, $|\widetilde{B}| = m|B|$, and $|V(\widetilde{X})| = m|V(X)|$. It also turns out that each edge in X is the image of exactly m edges in $\widetilde{X}$. To see this note that an edge e in X corresponds to a pair of cosets Hg_1 and Hg_2 such that $Hg_1 = Hg_1t$ for some $t \in T$. This means that for every $h \in H$ there is an $h' \in H$ such that $hg_1 = h'g_2t$. Each pair hg_1, $h'g_2$ determines an edge, $\tilde{e}$ in $\widetilde{X}$ such that $\Pi(\tilde{e}) = e$ and there are exactly m of them. From this it follows that:

$$|E(\widetilde{A}, \widetilde{B})| = m|E(A, B)|,$$

and therefore,

$$\frac{|E(\widetilde{A}, \widetilde{B})||V(\widetilde{X})|}{|\widetilde{A}||\widetilde{B}|} = \frac{|E(A, B)||V(X)|}{|A||B|}.$$

Since partitions of the form $\widetilde{A} \uplus \widetilde{B}$ are only *some* of the possible partitions of $V(\widetilde{X})$, taking infimums on both sides over all partitions of $V(\widetilde{X})$ and $V(X)$ respectively gives

$$\epsilon(\widetilde{X}) \le \epsilon(X).$$

From our definition of an expander, $\epsilon(\widetilde{X})$ is uniformly bounded below by a positive constant for all $\widetilde{X}$ in our family of Cayley graphs. Hence, $\epsilon(X)$ is also bounded below by this same constant for all X in our family of Schreier graphs. □ We are now in a position to prove our theorem.

PROOF OF THE THEOREM: Given $\mathcal{F} = X(S_n, T_n)$ as in the Theorem, let $\mathcal{F}' = \{X(H_n\backslash S_n, T_n)\}$ be the family of Schreier graphs corresponding to the subgroups $H_n = S_2 \times S_{n-2} \subseteq S_n$. By the Lemma it suffices to show that $\mathcal{F}'$ is not an expanding family. Let $X(H_n\backslash S_n, T_n)$ and suppose that $T_n = \{t_1, t_2, ..., t_k\}$ is a finite set of reversals in S_n. Then we identify $V(X)$ with the set $\{\{i, j\} \mid i, j \in \{1, 2, ..., n\}, i \neq j\}$ and note that edges connect $\{i, j\}$ to $\{i', j'\}$ whenever $i' = t(i)$ and $j' = t(j)$ for some $t \in T_n$. Let $A = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, ..., \{n-1, n\}\}$ and let $B = V(X) - A$. Note that the reversal $t = [i, j]$ maps all elements of A back to A, except possibly for $\{i-1, i\}$ and $\{j, j+1\}$. Thus each reversal in T_n contributes at most 2 edges to the set $E(A, B)$. This means $|E(A, B)| \le 2k$ so,

$$\frac{|E(A, B)|}{|A||B|}|V(X)| \le \frac{2k}{(n-1)[\binom{n}{2} - (n-1)]}\binom{n}{2}$$

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which simplifies to

$$\frac{|E(A,B)|}{|A||B|}|V(X)| \leq \frac{2kn}{(n-1)(n-2)}.$$

Since the expression on the right goes to zero (note k is fixed) as $n \to \infty$, $\mathcal{F}'$ cannot be an expander. Thus, by Lemma 4, $\mathcal{F}$ is not an expander. □ Note that even letting $|T_n|$ grow like $o(n)$, we still cannot obtain a uniform lower bound on expansion constants since we would simply change the right hand side of our inequality to

$$\frac{2n * o(n)}{(n-1)(n-2)},$$

which still goes to zero as $n \to \infty$. A natural question then is exactly what *growth rate* is needed. For example, can we form an expanding family by allowing the growth of $|T_n|$ to be of order $O(n)$? Results of this type are already known for the case when T_n is a set of transpositions [2] or T_n is a union of conjugacy classes [6].

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REFERENCES

[1] R. Bacher *Valeur Propre Minimale du Laplacien de Coxeter pour le Groupe Symétrique.* Journal of Algebra *167* (1994), 460–472.

[2] J. Friedman *On Cayley Graphs on the Symmetric Group Generated by Transpositions.* Combinatorica *20* (2000), 505–519.

[3] M. Klawe *Nonexistence of One-dimensional Expanding Graphs.* Symposium on the Foundations of Computer Science, Nashville, TN 1981.

[4] A. Lubotzky Discrete Groups, Expanding Graphs, and Invariant Measures. Progess in Mathematics, Vol. 125, Birkhuser, 1994.

[5] A. Lubotzky, R. Phillips, and P. Sarnak *Ramanujan Graphs.* Combinatorica *8* (1988), 261–277.

[6] Y. Roichman *Expansion Properties of Cayley graphs of the Alternating Groups.* Journal of Combinatorial Theory Ser A *79* (1997), 281–297.

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