# Contents







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# Introduction

The content of this work is meant to more than cover a one semester introduction to Abstract Algebra via Group Theory for mathematics students in a liberal arts setting. With that in mind, I have made the unorthodox decision to write this book in a much more familiar style than one would typically (read: effectively always) see in a mathematics journal article or textbook. Not only is there heavy usage of the familiar "you" rather than addressing the more formal and mysterious "reader," but there are also many parenthetical notes (typically in blue like this) and warnings in red meant to help clarify difficult portions of the text, anticipate common errors, or offer alternative ways of interpreting the definitions, examples, and results. The hope is that making the text easier to read and more colloquial will help encourage all of you (the students or independent learners) to read it!

The overarching goal of the book is two-fold. First, I would like you all to become familiar with the notion of groups and to gain the interest and motivation to look for groups everywhere. Essentially, any time you find a set that is closed under some nice operation I would like you to wonder whether that set (together with the operation) forms a group. Groups are somewhat fundamental objects that appear all over the place. Not just in the abstract realm of mathematics, but also in more exotic places like molecular biology and physics and probably many more places where they have yet to be noticed. One of the major aims (and strengths) of mathematics is to find generalizations that allow us to see the connections between many different-looking topics. Once the topics have been tied together through a common generalization, then the tools from each discipline become available to the other. I would love to see each of you go out into the world and find more of these kinds of connections!

With that goal in mind, the second aim really piggy-backs on the first – as we begin to understand groups, a very natural question to ask is: "What kinds of groups actually exist?" What does it mean for two groups (that look different) to be the same? In other words, suppose that – after working through this book and perhaps a related course – you find yourself searching for groups in your future area of work. One day, while working and searching you discover some group "in the wild" so to speak. How will you know which group it is? Or, more pertinently, how will you be able to use your knowledge of groups gained here to help you gain information about your wild group? If you can successfully identify the group you've found with another one that you already understand, then you'll be able to take everything that you've learned here and apply it to your new setting. So we'd like to both gain knowledge of many different common examples of groups and also learn how to relate two groups to one another via maps between them.

In order to improve your ability to recognize wild groups, a significant portion of the later material in this book is aimed toward developing tools which allow us to classify finite groups under different conditions. For one, there is a complete classification of all finite Abelian groups (which are special groups where the operation is commutative). We will also classify more general groups in special cases (essentially all groups of sizes 1 through 17) and will cover some infinite collections as well (groups of size $p$, $p^2$, or $pq$ for primes $p$ and $q$). Finally, for the rest of our abstract finite groups, I provide a foundation and some nice tools (the Sylow Theorems among others) for studying those groups and moving towards classification/identification.



### Changes to the Second Edition

After having run my own undergraduate level Abstract Algebra course a couple times with the first edition, I took student feedback I had collected and decided it was time for a second edition. The second edition provides a massive addition of material on top of the first, with 187 additional exercises throughout (taking the total number to 300), and spanning an extra 75 pages (even after I reduced the font size slightly to try to cut down on the size and reduce costs). While most students were quite happy with a majority of the first edition and praised its readability, they did point to later chapters – specifically Chapters 4 and 5 – as being light on examples. They were right. So one of the first goals of the update was to add new examples, exercises, and additional explanations to those chapters as well as the rest of the book.

As I continued to expand and improve those sections, I began to think about other closely related topics – topics that we do not regularly make it to in our course – that students might be interested in and that would tie in to our goal of classifying small groups. This prompted me to add new sections on products of subgroups (and how they relate to group decompositions) and the Fundamental Theorem of Finite Abelian groups, and an entirely new chapter on group actions, conjugation, Sylow subgroups, and the Sylow theorems. This additional material also strongly motivated the change to introducing equivalence relations back in Chapter 1 as it became worth it (in my opinion) to use that language once we had more than one application to deal with. Previously I proved nice facts about cosets (for example, that they partition the group) without ever introducing equivalence relations. This time around, I definitely make use of that concept (which shortens the proof and gives us another context in which to think about those results).

In a similar way, the addition of these extra topics necessitated small additions in other sections to support the development of the new results. Conversely, the new results also made small changes to some of the arguments later in the text. I believe that all of these changes have ultimately benefited the book by helping to clarify some tricky concepts and to extend the reach of this book to more challenging (and interesting!) parts of group theory. Of course, there are always more things that could be added. Even with all of the additions, I tried my best to stick to material that I felt would be reasonable for undergraduates in a liberal arts setting or to someone completely new to this material trying to learn it on their own (or using this as a supplement for a more difficult/dense/advanced text or course).

### Topics Covered

The text begins with a significant introductory chapter in which we cover many topics that you may have seen before (introductory set theory, number theory, induction, maps between sets, and equivalence relations) in the hopes that all of you will approach the group theoretic sections with a similar foundation. In Chapter 2 I introduce the notion of groups via examples which you should already be familiar with (although probably not from this point of view) before moving to more exotic examples with which you are probably less comfortable. In Section 2.3 specifically, we'll look at groups of symmetries of geometric objects. In many ways, this is where group theory began – with a goal of finding ways to "act on" other objects in nice ways. In fact, for some mathematicians, groups are only as interesting as they are useful in acting as "symmetries" on something else. Together we'll prove many nice properties that all groups share (using only the group axioms to justify those results) before starting to delve into the properties of individual elements and nice subsets (called subgroups).

Chapter 3 contains some important examples of infinite families of groups – the cyclic groups and the symmetric groups. As we'll see later in the book, cyclic groups provide the basic building blocks for all finite Abelian groups. The symmetric groups are also quite fundamental, as Cayley's Theorem demonstrates that every finite group can be found living inside a symmetric group of sufficiently large size. Once again, this is a way of thinking about all finite groups as really being collections of symmetries of some object. We round off this chapter by looking at a way of combining groups to get new ones via direct sums. Conversely, later in the book we will return to direct sums as a way of decomposing large groups into sums of smaller ones.

Chapter 4 discusses subgroups in further depth. Specifically introducing (set) products of subgroups, cosets of subgroups, and normal subgroups with an aim toward proving Lagrange's Theorem and defining quotient groups. Quotient groups are effectively a generalization of the notion of modular arithmetic to groups other than the group of integers $\mathbb{Z}$ as they allow us to "glue" elements of a group together in such a way that we can still make use of the original group operation. (Just like how we glue 12 and 0 together, and 13 and 1 together, etc. when telling time in "clock arithmetic.") It then follows those topics up by looking more closely at when a product of two subgroups is again a subgroup and developing a beautiful way to decompose groups in terms of products of some particularly nice subgroups (perhaps most interestingly, in terms of $p$-subgroups).

Chapter 5 then circles back to maps between sets with the additional motivation that we would like those maps to respect groups structures. Maps that do so are called group homomorphisms and they allow us to begin seeing connections between groups that "look" very different. The notion of isomorphisms (group homomorphisms that are also bijections) will then allow us to actually identify groups as being effectively "the same" and will allow us to begin classifying small finite groups. In the special case when a isomorphism is defined from a group back to itself, we call such maps automorphisms and the set of all such automorphisms actually forms a group in its own right. We spend a little bit of time studying and exploring the automorphism groups of some of our most familiar groups. This chapter continues with some nice structure theorems (the correspondence theorem and the isomorphism theorems) relating groups (and their quotients) to their images under homomorphisms. With these tools in hand we will then prove the Fundamental Theorem of Finite Abelian Groups, which classifies all finite Abelian groups as direct sums of cyclic groups. Finally, we'll finish off this chapter with Cayley's Theorem which gives us a way of understanding all finite groups as subgroups of the family of symmetric groups.

In Chapter 6, which is completely new to the second edition, we'll explore group actions, which are ways to have groups interact with other objects via their underlying sets. From my perspective, this would serve as a nice introduction to Representation Theory as well if you focus on the specific setting when groups act on vector spaces. Next, we focus more closely on the action of a group on itself via conjugation which has nice connections to earlier material in the book. In addition, the conjugation action is crucial to the proofs of Sylow's theorems which give fundamental results about nice $p$-subgroups (called $p$-Sylow subgroups – they are also introduced in this chapter), which must appear in all finite groups (whether they are Abelian or not). Throughout both Chapters 5 and 6 I will also make use of the techniques we develop to classify groups of small orders.

## Designing a Course

In the two years when I ran my course using the first edition textbook, I set my goal for the course as making it to the isomorphism theorems and Cayley's theorem, with classification of small groups up to order 7 (and groups of prime order) sprinkled in along the way. Towards that end, I covered parts of Chapter 1 as needed for the particular cohort of students (almost always the topics of introductory set theory, maps between sets, and induction). We then covered essentially all of Chapter 2 and all of Chapter 3 as we built up our familiarity with groups and group structures and also developed a bank of nice examples. Most of the results in Chapter 2 are heavily used throughout the rest of the book. Direct Sums specifically, Section 3.3 could be skipped if you do not intend on discussing the decomposition of groups (something we haven't covered in my course before, but that I hope to get to now with the second edition textbook)



In Chapter 4 we studied cosets and normal subgroups with an eye towards developing the notion of quotient groups. The old version of this chapter did not contain much about products of subsets or subgroups (outside of the special case of cosets) and had literally no treatment of conjugation or conjugacy classes for its own sake. The material about the normalizer is also new to the second edition, thus we did not cover it previously. These topics/results could certainly be skipped if you are not planning on making it to Chapter 6. The commutator subgroup and related results are also new and unnecessary for anything that follows. Perhaps most importantly, Section 4.4 is completely new and is only necessary if you plan to reach group decompositions, the Fundamental Theorem of Finite Abelian Groups, or the Sylow Theorems (also all new).

Chapter 5 introduces group homomorphisms and isomorphisms, the latter of which gives us a way to precisely determine when two groups that look different are really "the same." The basic results about group homomorphisms/isomorphisms and the more fundamental isomorphism theorems (at least the Correspondence Theorem and the First Isomorphism Theorem) are vital to the classification discussion. Automorphism groups are interesting – and I usually cover them – but they could probably be skipped in favor of other topics without missing much in the main story line. Section 5.4 with the Fundamental Theorem of Finite Abelian Groups is new and unnecessary for Cayley's theorem (my old stopping point), but I believe it is a beautiful part of the classification story that I definitely intend to cover from now on. Cayley's theorem, from Section 5.5, is a beautiful result with strong ties to the history of group theory, but it can certainly be skipped without losing the main thread.

If your class moves faster than mine, then you may wish to continue on into Chapter 6. You can safely cover group actions without any of the extra material needed for conjugation or the Sylow theorems and this can be a nice topic to end on. (I've done this at least a couple of times in the past – before I wrote my book.) It gives a nice lead in to Representation Theory if you focus on the actions of groups on vector spaces for example. It would be pretty easy to add the conjugation action from Section 6.2 – as a particularly nice example of groups acting on themselves – by either introducing conjugation earlier (as it appears in Chapter 4 for example) or for the first time at this point. To be able to say anything interesting about Sylow subgroups and the Sylow theorems you will need to have covered most of the rest of the text (definitely group actions and conjugation). To get to the applications of Sylow theorems it would also be nice for you to have covered the decomposition story line as well, since one of the main applications of the Sylow theorems is in helping to classify/decompose groups. In addition, one of the wonderful results that stems from the Sylow theorems is that you can decompose groups in terms of their Sylow subgroups whenever those subgroups are all normal.